

# COLLINEAR TRIPLES IN PERMUTATIONS

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ABSTRACT. Let  $\alpha : \mathbb{F}_q \rightarrow \mathbb{F}_q$  be a permutation and  $\Psi(\alpha)$  be the number of collinear triples in the graph of  $\alpha$ , where  $\mathbb{F}_q$  denotes a finite field of  $q$  elements. When  $q$  is odd Cooper and Solymosi once proved  $\Psi(\alpha) \geq (q-1)/4$  and conjectured the sharp bound should be  $\Psi(\alpha) \geq (q-1)/2$ . In this note we indicate that the Cooper-Solymosi conjecture is true.

## 1. THE MAIN RESULT AND ITS PROOF

Let  $\mathbb{F}_q$  be the finite field of  $q$  elements with  $q$  odd. Let  $\alpha : \mathbb{F}_q \rightarrow \mathbb{F}_q$  be a permutation and  $\Psi(\alpha)$  be the number of collinear triples in

$$G_\alpha = \{(i, \alpha(i)) : i \in \mathbb{F}_q\},$$

the graph of  $\alpha$ . Cooper and Solymosi [3] first (see also [6]) obtained the lower bound

$$(1.1) \quad \Psi(\alpha) \geq \frac{q-1}{4},$$

and conjectured the best one should be

$$(1.2) \quad \Psi(\alpha) \geq \frac{q-1}{2}.$$

Later Cooper [2] showed that the problem of counting collinear triples in a permutation and the finite plane Kakeya problem are intimately connected, and improved (1.1) into

$$(1.3) \quad \Psi(\alpha) \geq \frac{5q-1}{14}.$$

This is the right way to solve the Cooper-Solymosi conjecture and we shall explain Cooper's idea in more detail in Section 3.

A subset in  $\mathbb{F}_q^2$  containing a line in each direction is called a Kakeya set. According to Cooper, given the permutation  $\alpha$ , one can construct a corresponding Kakeya set

$$K_\alpha = \bigcup_{i \in \mathbb{F}_q} L(i, (0, \alpha(i))),$$

where  $L(s, x)$  denotes the line in  $\mathbb{F}_q^2$  through  $x$  with slope  $s$ . Let  $\Gamma_\alpha$  be the hypergraph on the vertex set  $G_\alpha$  whose edges are the maximal collinear subsets of  $G_\alpha$ , and write  $\|\Gamma_\alpha\|$  for the quantity

$$\sum_{e \in E(\Gamma_\alpha)} \binom{|e|-1}{2}.$$

Then by applying an incidence formula of Faber [4], Cooper successfully showed that

$$(1.4) \quad \sharp K_\alpha = \frac{q(q+1)}{2} + \|\Gamma_\alpha\|.$$

To confirm (1.2), by considering

$$\Psi(\alpha) = \sum_{e \in E(\Gamma_\alpha)} \binom{|e|}{3} \geq \sum_{e \in E(\Gamma_\alpha)} \binom{|e|-1}{2} = \|\Gamma_\alpha\|,$$

it suffices to prove

$$\sharp K_\alpha \geq \frac{q(q+1)}{2} + \frac{q-1}{2}.$$

So we are led to caring about the size of the finite plane Kakeya sets.

The finite field Kakeya problem posed by Wolff in his influential survey [7] asks for the smallest size a Kakeya set can have. Coincidentally, for any Kakeya set  $K \subset \mathbb{F}_q^2$  Faber [4] once proved the bound

$$\sharp K \geq \frac{q(q+1)}{2} + \frac{q}{3},$$

and conjectured the sharp one should be

$$(1.5) \quad \sharp K \geq \frac{q(q+1)}{2} + \frac{q-1}{2}.$$

Recently, this problem was solved by Blokhuis and Mazzocca [1], see also [2] for other medium bound. As an immediate corollary, the Cooper-Solymosi conjecture (1.2) turns out to be true.

To make this note more self-contained, the incidence formula of Faber (a new proof) and the connection between the permutations and the finite plane Kakeya sets discovered by Cooper (with more details) will be discussed in the next two sections. We refer the reader [1] for the original proof of the Faber conjecture (1.5).

## 2. THE INCIDENCE FORMULA OF FABER

Suppose  $K$  is a minimal Kakeya set in  $\mathbb{F}_q^2$ . Clearly, we may assume  $K$  is of the form

$$K = \bigcup_{s \in PG(1,q)} L(s, x^{(s)}).$$

Let  $\mu_x$  be the number of these lines passing through  $x \in \mathbb{F}_q^2$ . Obviously one has

$$\mu_x = \sum_{s \in PG(1,q)} \chi_{L_s}(x),$$

where we let  $\chi_A$  denote the characteristic function of  $A \subset \mathbb{F}_q^2$  and write  $L_s = L(s, x^{(s)})$  for simplicity. The incidence formula of Faber [4] says that

$$(2.1) \quad \sharp K = \frac{q(q+1)}{2} + \sum_{x \in K} \binom{\mu_x - 1}{2}.$$

In the following we will give a succinct proof of the Faber formula. As we know, two lines with different slopes intersect at one point. Hence considering

$$\begin{aligned}
(2.2) \quad \sum_{x \in K} \mu_x &= \sum_{x \in K} \sum_{s \in PG(1, q)} \chi_{L_s}(x) \\
&= \sum_{s \in PG(1, q)} \sum_{x \in K} \chi_{L_s}(x) \\
&= q(q+1), \\
(2.3) \quad \sum_{x \in K} \mu_x^2 &= \sum_{x \in K} \sum_{i \in PG(1, q)} \sum_{j \in PG(1, q)} \chi_{L_i}(x) \chi_{L_j}(x) \\
&= \sum_{i \in PG(1, q)} \sum_{j \in PG(1, q)} \sum_{x \in K} \chi_{L_i}(x) \chi_{L_j}(x) \\
&= \sum_{i \in PG(1, q)} \sum_{j \in PG(1, q)} \#(L_i \cap L_j) \\
&= 2q(q+1),
\end{aligned}$$

it follows that

$$\begin{aligned}
\sum_{x \in K} \frac{(\mu_x - 1)(\mu_x - 2)}{2} &= \frac{\sum_{x \in K} \mu_x^2}{2} - \frac{3 \sum_{x \in K} \mu_x}{2} + \#K \\
&= \frac{2q(q+1)}{2} - \frac{3q(q+1)}{2} + \#K \\
&= \#K - \frac{q(q+1)}{2}.
\end{aligned}$$

Finally we indicate that (2.2) and (2.3) already appeared in [5].

### 3. THE COLLINEAR TUPLE HYPERGRAPHS AND THE FINITE PLANE KAKEYA SETS

As before, given the permutation  $\alpha$ , one can construct a corresponding Kakeya set

$$K_\alpha = \bigcup_{i \in \mathbb{F}_q} L(i, (0, \alpha(i))).$$

Let  $x = (x_1, x_2) \in \mathbb{F}_q^2$  be any point satisfying  $\mu_x \geq 3$ . Since  $\alpha$  is a permutation,  $x_1 > 0$ . We assume that  $x$  lies in the lines

$$L(i_k, (0, \alpha(i_k))) \quad (k = 1, 2, \dots, \mu_x),$$

which means

$$\frac{x_2 - \alpha(i_k)}{x_1 - 0} = i_k.$$

Hence for all  $1 \leq j < k \leq \mu_x$ ,

$$\frac{\alpha(i_k) - \alpha(i_j)}{i_k - i_j} = -x_1.$$

In another words, the set

$$E_x \doteq \{(i_k, \alpha(i_k))\}_{k=1}^{\mu_x}$$

is collinear. In fact,

$$E_x \subset L(-x_1, (i_1, \alpha(i_1)))$$

is a maximal collinear subset of  $G_\alpha$ . For if

$$\frac{\alpha(k) - \alpha(i_1)}{k - i_1} = -x_1$$

holds for some  $k \neq i_1$ , then

$$\alpha(k) = \alpha(i_1) - (k - i_1)x_1 = x_2 - kx_1,$$

which means  $x$  lies in the line  $L(k, (0, \alpha(k)))$ . Thus  $k = i_j$  holds for some  $2 \leq j \leq \mu_x$  and  $E_x$  is a maximal collinear subset of  $G_\alpha$ .

On the other hand, suppose  $\{(i_t, \alpha(i_t))\}_{t=1}^\gamma$  is a maximal collinear subset of  $G_\alpha$ , where  $\gamma \geq 3$ . Define

$$z_1 = \frac{\alpha(i_2) - \alpha(i_1)}{i_1 - i_2}$$

and

$$z_2 = \alpha(i_1) + z_1 i_1,$$

then it is easy to verify that

$$(z_1, z_2) \in L(i_t, \alpha(i_t))$$

holds for  $t = 1, 2, \dots, \gamma$ .

In summary, the point  $x$  in  $K_\alpha$  with  $\mu_x \geq 3$  corresponds to a collinear  $\mu_x$ -tuple in  $G_\alpha$ , and vice visa. Consequently,

$$(3.1) \quad \sum_{x \in K_\alpha} \binom{\mu_x - 1}{2} = \sum_{e \in E(\Gamma_\alpha)} \binom{|e| - 1}{2}.$$

Combining (3.1) with the Faber formula (2.1) yields (1.4).

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